Multiplicative properties of de Rham-Witt

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Preamble: The bulk of this talk will be classical (going back to 1983). The last part is based on work in progress, in fact so much in progress that I don't have nearly as much to say about it as I'd like to. Also, this is a hard topic to give a satisfying talk on because a disproportionate amount of its complexity lies in the definitions of the objects and categories I'm talking about. It can be hard to keep all the moving parts in your head if it's not something you think about regularly. Please stop me and ask if you get confused about anything.

1 Review of the de Rham-Witt complex

Let k be a perfect field of characteristic p, $W = W(k)$, and $\sigma : W \to W$ the Witt vector Frobenius. Let X/k be a variety, which in this talk (for simplicity) will always be smooth and usually proper.

The de Rham-Witt complex of X/k, first constructed by Illusie in 1979, is designed to lift the de Rham complex $\Omega^\bullet_{X/k}$ to characteristic 0, and thereby to compute crystalline cohomology. It is defined as the initial object in a rather complicated category. Rather than giving its full definition, I will just outline what kinds of structure it has, and some of the key conditions we impose. It contains the data:

$$\begin{tikzcd}
\vdots \arrow[d, "R"] & \vdots \arrow[d, "R"] & \\
W_2\mathcal{O}_X \arrow[r, "d"] \arrow[d, "R"] & W_2\Omega^1_X \arrow[r, "d"] \arrow[d, "R"] & \cdots \\
W_1\mathcal{O}_X \arrow[r, "d"] & W_1\Omega^1_X \arrow[r, "d"] & \cdots
\end{tikzcd}$$

Here each $W_n\Omega^i_X$ is a sheaf of $W_n\mathcal{O}_X$-modules, with $W_n(k)$-linear differentials and vertical quotient maps. (The bottom row is just the de Rham complex of X, and the leftmost column is the sheaf of Witt vectors of $\mathcal{O}_X$.) Additionally, each row has a multiplication map making it a cdga. Finally, each column has maps F going down and V going up, satisfying the following relations:

*Notes for a talk in Stanford's student algebraic geometry seminar. Main reference: Ekedahl, "On the multiplicative properties of the de Rham-Witt complex II".

(a) $FV = VF = p$

(b) $dF = pFd$, $Vd = pdV$, $FdV = d$,

(c) $F(a) = \sigma(a)$ and $V(a) = p\sigma^{-1}(a)$ for $a \in W$,

(d) $F(xy) = F(x) \cdot F(y)$,

(e) $V(F(x) \cdot y) = x \cdot V(y)$,

and a few others.

The complex $W\Omega^\bullet_X$ is defined as $\lim_{\leftarrow} W_n\Omega^\bullet_X$. The F, V, and d operators and the multiplication map pass to the inverse limit, and they have the same relations as above. Given $W\Omega^\bullet_X$ with all of these operators, we can recover $W_n\Omega^\bullet_X$ as its quotient by the images of V^n and dV^n. In practice, we pass between $W\Omega^\bullet_X$ and $(W_n\Omega^\bullet_X)_n$ more or less freely, but one must be somewhat cautious about what operations do and don't commute with the limit.

Remark: Under our smoothness hypotheses, $W\Omega^\bullet_X$ turns out to be p-torsion-free. Then each of the relations in (b) above is equivalent to saying that the map φ defined by p^iF on $W\Omega^i$ commutes with d. This gives rise to the semilinear Frobenius operator φ on crystalline cohomology.

Theorem: The hypercohomology of the de Rham-Witt complex computes crystalline cohomology. More precisely, we have isomorphisms

$$Ru_{n*}(\mathcal{O}_{X/W_n,\mathrm{cris}}) \cong W_n\Omega^\bullet_X \tag{1}$$
$$Ru_*(\mathcal{O}_{X/W,\mathrm{cris}}) \cong W\Omega^\bullet_X \tag{2}$$

in $D(\mathrm{Ab}(X))$, where $u_n : (X/W_n)_{\mathrm{cris}} \to X_{\mathrm{Zar}}$ and $u : (X/W)_{\mathrm{cris}} \to X_{\mathrm{Zar}}$ are the usual morphisms of sites. After applying $R^i\Gamma$, these become isomorphisms

$$H^*_{\mathrm{cris}}(X/W_n) \cong \mathbb{H}^*(W_n\Omega^\bullet_X) \tag{3}$$
$$H^*_{\mathrm{cris}}(X/W) \cong \mathbb{H}^*(W\Omega^\bullet_X). \tag{4}$$

2 Example: $\mathbb{G}_m$

Let's write down the de Rham-Witt complex of $X = \mathbb{G}_m = \operatorname{Spec} A$, $A = k[t^{\pm 1}]$. This is something that can be done entirely by hand, as Illusie does (and as I have done with Bhatt-Lurie-Mathew's construction). But the full calculation is a bit tedious, so I'll just give you the result along with a rough plausibility argument. I'll write down the global sections of $W\Omega^i_X$, from which one can calculate the global sections of $W_n\Omega^i_X$, and these determine the sheaf because it is quasicoherent when viewed as a sheaf on the affine scheme W_nX.

Let's first write down a reasonable guess for what $W(A)$ might look like. It should contain $W[t^{\pm 1}]$, where t is the Teichmüller lift of $t \in A$. This ring has an obvious lift of Frobenius, given by $t \mapsto t^p$ and the Witt vector Frobenius on coefficients. But the Verschiebung should send t^α to $pt^{\alpha/p}$ in order to get $FV = VF = p$. So we must adjoin $p^n t^{m/p^n}$ for all $m \in \mathbb{Z}$ and $n \geq 0$. The resulting ring is almost right, but isn't complete. In fact, $W(A)$ is equal to the V-adic completion (equivalently, coefficient-wise p-adic completion) of this ring:

$$W(A) = (W[t^{\pm 1}, p^n t^{m/p^n} : m \in \mathbb{Z}, n \geq 0])^{\widehat{}}_V \hookleftarrow W[t^{\pm 1}, p^n t^{m/p^n}] \hookleftarrow W[t^{\pm 1}] \tag{5}$$

$$F(t^\alpha) = t^{p\alpha}, \qquad \text{(and } \sigma \text{ on coefficients)} \tag{6}$$

$$V(t^\alpha) = pt^{\alpha/p}. \tag{7}$$

This is (the global sections of) $W\mathcal{O}_X$. As for $W\Omega^1_X$, we have:

$$W\Omega^1_X(X) = W[t^{\pm 1/p^\infty}]^{\widehat{}} \cdot \frac{dt}{t}, \tag{8}$$

$$F\left(t^\alpha \frac{dt}{t}\right) = t^{p\alpha} \frac{dt}{t}, \tag{9}$$

$$V\left(t^\alpha \frac{dt}{t}\right) = pt^{\alpha/p} \frac{dt}{t}. \tag{10}$$

(We choose dt/t as our basis to make the formulas for F and V look nicer.) The differential $d : W\mathcal{O}_X \to W\Omega^1_X$ sends t^α to $\alpha t^\alpha dt/t$, as one would expect. (The completion in degree 1 is a little more subtle than that in degree 0, since we are completing with respect to the image of dV^n as well as V^n. This is needed for the differential to be defined. But this won't matter for us.)

Remark: There is a similar description of $W\Omega^\bullet_{\mathbb{A}^1}$, where we only allow nonnegative powers of t in degree 0 and positive powers in degree 1. This determines $W\Omega^\bullet_{\mathbb{P}^1}$.

Exercise: compute the sheaf cohomology of each $W\Omega^i_{\mathbb{P}^1}$ using the obvious Čech cover. Then use the slope spectral sequence $H^j(W\Omega^i_X) \implies H^{i+j}_{\text{cris}}(X/W)$ to compute crystalline cohomology.

3 Ekedahl's work

Next, let me summarize (part 2 of) Ekedahl's thesis, "Multiplicative properties of the de Rham-Witt complex". The goal is to prove Künneth and duality formulas for crystalline cohomology as generally and formally as possible, with most of the work happening on the de Rham-Witt complex.

For concreteness, let's very briefly sketch the classical proof of the Künneth formula for $X, Y/k$ smooth proper. The cup product gives a morphism

$$R\Gamma_{\text{cris}}(X/W) \otimes^L_W R\Gamma_{\text{cris}}(Y/W) \to R\Gamma_{\text{cris}}(X \times_k Y/W) \tag{11}$$

in $D(W)$. We claim that this is an isomorphism.

Fact: given that both sides have finitely generated cohomology, it suffices to check that it is a quasi-isomorphism after $\otimes^L_W k$. This reduces the problem to the Künneth formula for de Rham cohomology.

The problem with this is that the hypothesis only holds when X and Y are smooth and proper. (Recall that $H^1_{\mathrm{cris}}(\mathbb{A}^1/W) \cong (\bigoplus_{n\geq 0} W/nW)^{\widehat{}}_p$.) This can be fixed by using the completed tensor product $\widehat{\otimes}^L_W$ instead of the usual one. One must then show that $\widehat{\otimes}^L_W$ agrees with $\otimes^L_W$ for complexes with finitely generated cohomology.

Ekedahl mimics this latter proof, but constructs the map at the level of $W\Omega^\bullet_X(U)$ for U affine. He views this as a module over a ring much larger than W, and defines a symmetric monoidal structure for such modules that is universal for the product relations stated earlier. Once he has built up (and derived) a suitable tensor formalism, the proof is formal. The result is an isomorphism of the form

$$Rf_*W\Omega^\bullet_X \widehat{*}^L_R Rg_*W\Omega^\bullet_Y = R(f\times g)_*W\Omega^\bullet_{X\times Y} \tag{12}$$

under very mild hypotheses, and one can then hope to compare $\widehat{*}^L_R$ to $\otimes^L_W$. (I will omit the statement of his duality theorem, as it would require discussing perfect complexes of R-modules.)

Definition: The Raynaud ring R is the (noncommutative graded) W-algebra generated by F and V in degree 0 and d in degree 1, with all the relations stated earlier. (This has a W-basis consisting of elements of the form F^n, V^n, F^nd, and dV^n.) Unless stated otherwise, all R-modules we discuss will be graded left modules.

Next we construct the operation on R-modules that will replace $\otimes_W$. This is defined as the product universal for the multiplicative relations satisfied by d, F, and V. Namely, for $M, N \in R-\mathrm{mod}$, we let $M *_R N$ be the R-module generated by symbols $m * n$, with the following relations:

- W-bilinearity,
- $d(m*n) = d(m)*n + (-1)^{|m|}m*dn$ for m homogeneous,
- $F(x*y) = F(x)*F(y)$,
- $V(F(x)*y) = x*V(y)$, and $V(x*F(y)) = V(x)*y$.

This makes R-mod a symmetric monoidal category with unit object W (where $F=1$ and $V=p$). In particular, $R*_R R$ is *not* isomorphic to R; it is free of countably infinite rank over R.

Remark: Roughly, $M*N$ looks like $M\otimes_W N$ with a Verschiebung adjoined. I'll give another way to think about it later.

Strictly speaking, we really care about complete R-modules and the completed star product $\widehat{*}_R$, where the completion is with respect to the topology defined by $(\mathrm{im}\,V^n + \mathrm{im}\,dV^n)_n$. But I won't have time to say much about this.

Ekedahl next constructs an internal Hom functor $\underline{\mathrm{Hom}}^!_R$ satisfying the usual Hom-tensor adjunction with $*_R$. This is more subtle than it sounds. It turns out what we need is the following:

$$\underline{\mathrm{Hom}}^!_R(M,N) = \underline{\mathrm{Hom}}_R(R *_R M, N). \tag{13}$$

Since R is noncommutative, this only obviously has the structure of a graded group (whose ith graded piece consists of degree-i maps). But R is also a right R-module, so $R *_R M$ is a bimodule (with right-multiplication acting only on R), and this extra structure gives the Hom-set the structure of a left R-module.

Most of the work in Ekedahl's paper consists of extending these functors to the (suitably bounded) derived category of R-modules, and checking that they have the expected properties.

4 Quick recap of BLM

Bhatt-Lurie-Mathew gives a new construction of the so-called *saturated de Rham-Witt complex* of an $\mathbb{F}_p$-scheme, which agrees with the classical one for smooth schemes. This is constructed affine-locally to satisfy a certain universal property, and the key categories involved are as follows.

Definition: a Dieudonné complex is a complex M^* of abelian groups equipped with an endomorphism $F : M^i \to M^i$ for each i such that $dF = pFd$.

Definition: a Dieudonné complex is *saturated* if it is p-torsion-free and the map $\varphi : M^* \to \eta_p M^*$ given by $p^i F$ in degree i is an isomorphism.

Exercise: a saturated Dieudonné complex has a unique operator V such that $FV = p$, and this also satisfies $VF = p$, $Vd = pdV$, $FdV = d$, etc.

Definition: a Dieudonné complex is *strict* if it is saturated and complete with respect to the topology defined by $(\operatorname{im} V^n + \operatorname{im} dV^n)_n$.

Then we have inclusions of categories $\mathbf{DC} \hookleftarrow \mathbf{DC}_{\text{sat}} \hookleftarrow \mathbf{DC}_{\text{str}}$, with left-adjoints Sat and $\mathcal{W}$.

Theorem (Bhatt-Lurie-Mathew): $\mathbf{DC}_{\text{str}}$ is equivalent to $\widehat{D(\mathbb{Z})}_p^{L\eta_p}$. So every object in the derived category of $\mathbb{Z}$-modules that "looks enough like crystalline cohomology" can be uniquely represented by a strict Dieudonné complex.

5 My project

Motivation: R-modules look a lot like saturated Dieudonné complexes, and complete R-modules look a lot like strict Dieudonné complexes. The main difference between the R-module world and the saturated world is the rigidity property of strict Dieudonné complexes mentioned above, which effectively allows us to turn quasi-isomorphisms into isomorphisms. The hope is that this will make $\mathbf{DC}_{\mathrm{str}}$ a good setting in which to rework Ekedahl's thesis, with the goal of generalizing it as far as possible (adding coefficients, working over a non-perfect base, ...).

Observation: The data of a Dieudonné complex is the same as the data of a graded left module over a particular graded ring, $R' = \mathbb{Z}\langle F, d\rangle/(d^2, dF - pFd)$. (Here we put F in degree 0 and d in degree 1.) It follows that R' itself is the free Dieudonné complex on one generator (in degree 0), and $\operatorname{Sat} R'$ is the free saturated Dieudonné complex on one generator. But $\operatorname{Sat} R' \otimes_{\mathbb{Z}} W$ is exactly the Raynaud ring R.[1]

Let me be more precise about the relationship between the various categories we are concerned with. We have the following diagram of various categories, with various adjoint pairs of functors. (I'll tell the $\mathbb{Z}$-linear story; note that $\operatorname{Sat} R'$ and $\mathcal{W}\operatorname{Sat} R'$ become R and $\widehat{R}$ upon tensoring up to W.)

$$\begin{tikzcd}
\mathbf{DC} \arrow[d, equal] & \mathbf{DC}_{\mathrm{sat}} \arrow[l] \arrow[d] & \mathbf{DC}_{\mathrm{str}} \arrow[l] \arrow[d] \\
R' - \mathrm{mod} & (\operatorname{Sat} R') - \mathrm{mod} \arrow[l] & (\mathcal{W}\operatorname{Sat} R') - \mathrm{mod}. \arrow[l]
\end{tikzcd}$$

The second and third vertical maps are *not* equivalences of categories. Example: let $M^* = \mathbb{Z}_p[\sqrt{p}]$, concentrated in degree 0, with $F = V = \sqrt{p}$. This is a $\mathcal{W}\operatorname{Sat} R'$-module, but it is not saturated, as $p^0 F : M^0 \to (\eta_p M)^0 = M^0$ is not an isomorphism.

Ekedahl's star product and internal Hom have analogues in all of these categories. In $\mathbf{DC}$, the analogue of $*_R$ is just the $\mathbb{Z}$-linear tensor product of complexes, equipped with F defined by $F(x \otimes y) = F(x) \otimes F(y)$ and d defined by the graded Leibniz rule. In $\mathbf{DC}_{\mathrm{sat}}$ and $\mathbf{DC}_{\mathrm{str}}$, the analogues are $\operatorname{Sat}(- \otimes -)$ and $\mathcal{W}\operatorname{Sat}(- \otimes -)$.

To construct an internal Hom functor in $\mathbf{DC}$, we imitate Ekedahl's construction in R-mod:

$$\underline{\mathrm{Hom}}^!_{\mathbf{DC}}(M, N) = \underline{\mathrm{Hom}}_{\mathbf{DC}}(R' \otimes_{\mathbb{Z}} M, N) \tag{14}$$

as a graded group, with Dieudonné complex structure coming from the right-R'-module structure ("right-Dieudonné complex structure") of R'. This satisfies the expected adjunction, and can be carried over to $\mathbf{DC}_{\mathrm{sat}}$ and $\mathbf{DC}_{\mathrm{str}}$. (TO DO: can it?)

The upward and rightward maps in this diagram are compatible with symmetric monoidal structures. The leftward ones aren't. In order to imitate Ekedahl's work, I will need to show that the downward maps are, at least after passing to W_1. (to do).

A bigger challenge: $\mathbf{DC}_{\mathrm{sat}}$ and $\mathbf{DC}_{\mathrm{str}}$ are tricky to work with. They're not abelian categories, so it will be harder to discuss derived functors from them.

[1]Remark: the construction of Bhatt-Lurie-Mathew is absolute; it makes no reference to the ground field, but recovers the classical de Rham-Witt complex of every variety that is smooth over some perfect field of characteristic p.